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MODELS FOR THE BUCKLING OF BARS ON AN ELASTIC BASE

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The present work deals with an $(m+1)$-parameter family of mathematical models describing the postbuckling behavior of a hinge-supported bar lying on a nonlinearly elastic base and loaded by an axial compressive force. Analytical expressions for the buckling modes and load–deflection dependence are given which are constructed using the perturbation method. Analysis is performed of the initial behavior of the system as a function of the parameters characterizing the base stiffness. It is noted that the postbuckling behavior of the system can be unstable. Contradictions are revealed that occur when some well-known models [1] of an elastic base are used to describe the buckling of a bar. By using catastrophe theory, models of the family are indicated that reflect the buckling process most adequately [2].

1. Statement of the Problem. We consider a hinge-supported bar of length L lying on a nonlinear-elastic base and loaded by an axial compressive force P whose direction remains unchanged upon deformation of the bar [3, 4]. The length L of the axial line of the bar is assumed to be invariable. We denote by l the distance between the bar ends. It is assumed that the bar axis can bend only in the plane. Let us investigate the postbuckling behavior of the bar–base system predicted by different models.

2. Family of Models. Let us write the expression for the total potential energy of the system in the form

$$U = -\frac{1}{2} EI \int_0^L æ^2 ds - P(L-l) + \int_0^L\int_0^w (c_1 w + c_3 w^3 + \ldots + c_{2m-1} w^{2m-1})(1 - w_s^2)^{\beta/2} dw ds, \qquad (2.1)$$

where EI is the flexural stiffness; $æ$ is the curvature of the bar axis; the parameters c_i take into account the linear ($i = 1$) and nonlinear (odd $i \geqslant 3$) components of the base stiffness [3]; s is the length of the bar axis arc. The function $w(s)$ ($0 \leqslant s \leqslant L$) describes [3–5] the deformed position of the bar and must satisfy the geometric boundary conditions of the problem:

$$w(0) = w(L) = w_{ss}(0) = w_{ss}(L) = 0. \qquad (2.2)$$

The factor $(1 - w_s^2)^{\beta/2}$ in expression (2.1) takes into account the geometric nonlinearity of the base, i.e., the change in the response of the base upon deflection of the bar [4]. Let us express the curvature $æ$ and the distance l in terms of the function $w(s)$ and substitute the expression into (2.1). The function $w(s)$ minimizing functional (2.1) satisfies the Euler equation, which can be written, with accuracy up to sixth-order terms containing the function $w(s)$ and its derivatives, in the form

$$EIw_{ssss} + EI(w_{ss}^2 + 4w_s w_{sss} + 4w_s^3 w_{sss} + 5w_s^2 w_{ss}^2) w_{ss} + P\left(1 + \frac{1}{2} w_s^2 + \frac{3}{8} w_s^4\right) w_{ss}$$

$$+ c_1 w \left(1 + \frac{\beta - 2}{2} w_s^2 + \frac{\beta(3\beta - 2)}{8} w_s^4\right) + c_3 w^3 \left(1 + \frac{\beta - 2}{2} w_s^2\right) = 0. \qquad (2.3)$$

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Solving Eq. (2.3) subject to boundary conditions (2.2) by the perturbation theory method, as in [3, 4], we obtain an expression for the initial buckling mode of the bar in terms of n semiwaves

$$w_n(s) = a_n \sin \frac{n\pi s}{L} + k_n a_n^3 \sin 3\frac{n\pi s}{L} + O(a_n^5) \tag{2.4}$$

[the magnitude a_n is approximately equal to the bar deflection amplitude (it is assumed that a $a_n/L \ll 1$), k_n is a variable that depends on n, c_1, c_3, and β but not than on s], and the load–deflection relationship is

$$P_n = EI\left(\frac{\pi}{L}\right)^2\left[n^2 + \frac{r_1}{n^2} + \frac{n^6 - n^2 r_1(3-\beta) + 6r_3}{8n^2}\left(\frac{\pi a_n}{L}\right)^2\right] + O(a_n^4), \tag{2.5}$$

where $r_1 = c_1(L/\pi)^4/EI$; $r_3 = c_3(L/\pi)^6/EI$. For $r_1 = n^2(n+1)^2$, apart from the symmetric equilibrium forms (2.4), asymmetric forms can exist, which are considered in detail in [3] for $\beta = 1$, $c_i = 0\,(i > 3)$ and in [4] for $c_i = 0\,(i > 1)$. Therefore, the case where $r_1 = n^2(n+1)^2$ is not discussed here.

3. Instability of Postbuckling Behavior. Analysis of expression (2.5) shows that the stability of the system's postbuckling behavior for the mode $w_n(s)$ is determined by the sign of the expression $n^6 - n^2 r_1(3 - \beta) + 6r_3$, which depends on three parameters: β, r_1, and r_3. Thus, for $r_i = 0$ we have the classical problem of the buckling of an elastic bar without a base, the postbuckling behavior of this system being stable for every mode. For $r_1 = 1$, $r_3 = -1$, and $\beta = 2$, which corresponds, with accuracy up to forth-order infinitesimal terms with respect to bar buckling, to the model in [6], we obtain $n^6 - n^2 - 6 < 0$ for $n = 1$, i.e., unstable initial postbuckling behavior in terms of one semiwave. The possibility of unstable postbuckling behavior in some classical models was shown in [4] and established experimentally in [7]. It is therefore natural to expect that in models that adequately describe [2] the buckling of a bar on an elastic base, the possibility of unstable postbuckling behavior cannot be denied. For instance, there is no point in choosing a model in which $\beta > 3$ and $r_i > 0$.

4. Potential Energies of Equilibrium Configurations. Let us now classify the various buckling modes for any fixed loading value P with the aid of the corresponding values of potential energy. Recal that examination of the case of $r_1 = n^2(n+1)^2$ involves no fundamental difficulties, and for some models it was carried out in [3, 4] and, therefore, is not given here. From (2.1), with accuracy up to fourth-order terms (inclusive) containing the function $w(s)$ and its derivatives, we find

$$U \approx \frac{1}{2}EI\int_0^L w_{ss}^2(1 + w_s^2)\,ds - P\int_0^L\left(\frac{1}{2}w_s^2 + \frac{1}{8}w_s^4\right)ds + \iint_{00}^{Lw} c_1 w\left(1 - \frac{\beta}{2}w_s^2\right)dw\,ds + \iint_{00}^{Lw} c_3 w^3\,dw\,ds, \tag{4.1}$$

which does not depend on c_i, $i \geqslant 5$. From (4.1) and the analytical representation of the buckling modes (2.4) it is clear that, in calculating the potential energy with accuracy up to a_n^4, it is sufficient to take into account in (2.4) a first-order term containing a_n. For the mode w_n we obtain

$$U_n \approx \frac{L}{64}\left(\frac{\pi}{L}\right)^6 EIa_n^4[-n^6 + n^2 r_1(3 - 7\beta) - 6r_3], \tag{4.2}$$

i.e., the function of the total potential energy of the ideal system that lost stability for the mode w_n has the form of a germ of a cusp catastrophe ($U_n > 0$), of a dual cusp ($U_n < 0$), or of a catastrophe that is more complicated than a cusp [8]. Since cusp and dual cusp catastrophes are stable, small perturbations in the model of a bar on an elastic base that take into account the imperfectness of the system (initial imperfections, eccentricity in loading) do not change the type of catastrophe and, hence, do not affect significantly expression (4.2).

5. Emerging Contradictions. Note that the potential energy (2.1) of the system having the undeflected shape $w \equiv 0$ is equal to zero. It is natural to regard the mathematical model of the bar–base system as inadequate to physical reality [2] if in the case where the load exceeds the critical value the total potential energy of the deflected state turns out to be higher than the potential energy of the undeflected system. For the models considered, this requirement involves the following prohibition: the function of the potential energy must not change the type of catastrophe from a dual cusp to a cusp. Therefore, to avoid any contradiction with the minimum potential energy principle one should choose only models for which the inequality

$$-n^6 + n^2 r_1(3 - 7\beta) - 6r_3 \leqslant 0, \tag{5.1}$$

holds, for instance, the model of a bar without an elastic base ($r_i = 0$) and the model considered in [6] with $r_1 = 1$, $r_3 = -1$, and $\beta = 2$ because $-n^6 - 11n^2 + 6 \leqslant 0$ for any natural n; choosing the model with $r_1 = 0$, we find from (5.1) the restriction $r_3 \geqslant -1/6$. Some classical models [1, 5], however, do not satisfy inequality (5.1) [4]. If the postbuckling behavior of the system is considered only in the vicinity of the first critical loading, it is necessary to fulfill the additional restrictions

$$(n-1)^2 n^2 < r_1 < n^2(n+1)^2, \tag{5.2}$$

which are used to determine the number of semiwaves of a sinusoid to whose shape the bar begins to buckle. For the model with $r_3 = -r_1 < 0$, we obtain from (5.1) and (5.2) a restriction on the parameter β, namely, $\beta \geqslant 5/4$, which is satisfied by the model of [6]. Dividing both sides of inequality (5.1) by n^6, passing to the limit in n, and taking into account (5.2), we have

$$2 - 7\beta \leqslant 6 \lim_{n\to\infty} (r_3/n^6)$$

and the restriction $\beta \geqslant 2/7$ for $r_3 = \text{const}$. If for the adequacy [2] of the selected model we take into account additionally, for at least one n, the necessity of fulfilling the inequality

$$n^6 - n^2 r_1(3 - \beta) + 6r_3 \leqslant 0, \tag{5.3}$$

which ensures the possibility of unstable postbuckling behavior (see Section 3), from (5.1) and (5.3) it follows that $\beta \geqslant 0$. The parameter β characterizes [4] the geometric nonlinearity of the elastic bar, and in [3] a detailed study is performed of the model ($\beta = 1$, $c_1 > 0$, and $c_3 \neq 0$), which is free of some imperfections inherent in the models of [1, 5]. Note that the main terms of expressions (2.5) and (4.2) are independent of r_i ($i > 3$); therefore, the choice of the parameters c_i ($i > 3$) does not affect significantly the description of the initial postbuckling behavior. However, in choosing the coefficients c_1, c_3, and β it is necessary to check the fullfilment of inequality (5.1) for any n, and of inequality (5.3) for at least one n.

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